Lecture 8   Convolution operators on the space of cusp forms

Terminology from functional analysis

H: Hilbert space $T: H \to H$ is <u>bounded</u> (or continuous)

if $\|Tv\| \le C\|v\|$ ($\exists C\ \forall v \in H$)

T: bounded $\rightsquigarrow$ T^*: adjoint $\langle Tv_1, v_2\rangle = \langle v_1, T^* v_2\rangle$
(another bounded operator)

<u>Defn</u> Let T: bounded. T is <u>compact</u> if
$\forall$ bounded sequences $v_j \in H$,
$\exists j_k$: subsequence s.t. Tv_{j_k} has a limit.

<u>Fact</u>: spectral theorem for compact self-adjoint operators $T: H \to H$.
$\exists$ orthonormal basis of eigenfunctions $v_j \in H$ for T.
$v_j \rightsquigarrow$ eigenvalue λ_j. We have $\lambda_j \to 0$ as $j \to \infty$,
i.e., $\forall \varepsilon > 0$, $\# \{|\lambda_j| \ge \varepsilon\}$ is finite.
In particular, $\forall \lambda \ne 0$, the eigenspace $H_\lambda := \{v \in H : Tv = \lambda v\}$ is finite-dimensional.

<u>Defn</u> T is <u>Hilbert-Schmidt</u> if $\|T\|_{HS}^2 = \sum_{i,j} |\langle Tv_j, v_i\rangle|^2 < \infty$
for some ONB (v_j) (equivalently, for any ONB).

<u>Defn</u> Given a bounded operator T, we define $|T| := \sqrt{T^* T}$.
Meaning: T^*T is a <u>positive</u> self-adjoint bounded. $\rightsquigarrow$ "positive square root"

<u>Defn</u> T: <u>trace class</u> $\iff$ $\operatorname{trace}(|T|) := \sum_j \langle |T| v_j, v_j\rangle < \infty$
for some (equivalently, any) ONB (v_j).

<u>Rmk</u> Suppose T is "diagonal": $\exists$ ONB v_j, $Tv_j = \lambda_j v_j$.
Then T: compact $\iff \lambda_j \to 0$ $(j \to \infty)$, T: Hilbert-Schmidt $\iff \sum |\lambda_j|^2 < \infty$
T: trace class $\iff \sum |\lambda_j| < \infty$.

In general, trace class $\Rightarrow$ HS $\Rightarrow$ compact.

Closed
$\downarrow$
$L_0^2 := L^2_{cusp} \subseteq L^2$
$:= \{$cuspidal elements of $L^2\}$

<u>Theorem</u> Let $\Gamma \backslash G = SL_n(\mathbb{Z}) \backslash SL_n(\mathbb{R})$ (so $\operatorname{vol}(\Gamma\backslash G) < \infty$).
Let $f \in C_c^\infty(G)$. Then the operator $T_f : L_0^2(\Gamma\backslash G) \to L_0^2(\Gamma\backslash G)$,
$T_f \varphi := \varphi * f$, is trace class (hence compact).
$: \Gamma\backslash G \ni x \mapsto \int_G \varphi(xg^{-1}) f(g)\, dg$

Proof sketch We first discuss compactness of T_f. Main point: $\forall \varphi \in L^2_0(\Gamma\backslash G)$,

$$(*) \qquad \|T_f \varphi\|_\infty \leq C(f)\|\varphi\|_{L^2}.$$

$(*)$ implies compactness, as follows.

$\forall D \in \mathcal{U}(\mathfrak{g})$, we have

$$D T_f = D(\varphi * f) = \varphi * Df,$$

so $\|D T_f \varphi\|_\infty \leq C(Df)\|\varphi\|_{L^2}$.

Thus if $\varphi_j \in L^2_0(\Gamma\backslash G)$: bounded sequence, then $\forall D$, $\|D T_f \varphi_j\|_\infty$: bounded.

$\Rightarrow T_f \varphi_j$: equicontinuous family of functions on $\Gamma\backslash G$

$\underset{\text{(Arzela-Ascoli)}}{\Rightarrow}$ $\exists$ subsequence $T_f\varphi_j$ that converges normally (uniformly on compacta)

$\underset{(*),\ \mathrm{vol}(\Gamma\backslash G)<\infty}{\Longrightarrow}$ $T_f \varphi_j$ converges in $L^2_0(\Gamma\backslash G)$. Hence $(*) \Rightarrow$ compactness.

Note φ : cuspidal $\Rightarrow$ $T_f \varphi$: cuspidal

$\Updownarrow$ $\qquad\qquad$ $\Updownarrow$

$\forall P \subsetneq G$, $P = MU$: $\int_{\Gamma_U\backslash U} \varphi(ux)\,du = 0$ $\forall x \in G$ $\qquad$ $\int_{\Gamma_U\backslash U} \left(\int_G \varphi(uxg^{-1}) f(g)\,dg \right) du$

$$= \int_G f(g) \left(\int_{\Gamma_U\backslash U} \varphi(uxg^{-1})\,du \right) dg$$

Proof of $(*)$ is similar to the arguments used to verify that φ : cusp form (:= cuspidal automorphic) $\Rightarrow$ φ : rapid decay, in particular bounded.

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compactness : ✓ $\qquad$ HS, $\qquad$ trace class : ?

$\forall x \in \Gamma\backslash G$, the functional $L^2_0 \ni \varphi \mapsto T_f\varphi(x)$ is bounded, hence $\exists k_x \in L^2_0$ s.t. $T_f \varphi(x) = \langle \varphi, k_x \rangle$.

Moreover, $\|k_x\| \leq C(f)$. check (Borel, SL_2, Thm 9.5)

$\Rightarrow \infty > \int_{\Gamma\backslash G} \|k_x\|^2 = \|T_f\|^2_{HS}$.

Lemma of Dixmier - Malliavin $\forall$ Lie group G, $\forall f \in C_c^\infty(G)$, we may write f as a finite sum of convolutions $f_1 * f_2$ $(f_j \in C_c^\infty(G))$

$\Rightarrow$ if T_f : HS $\forall f$, then T_f is also a finite sum of compositions $T_{f_1} T_{f_2}$ of HS operators, hence trace class.

G acts on $L^2_0(\Gamma\backslash G) =: L^2_0$ by right translation.

This defines a continuous map $G \to \{\text{bounded operators on } L^2_0\}$.

Defn A closed invariant subspace V of L^2_0 is irreducible if there are no nonzero proper invariant subspaces.

invariant: $G \cdot V \subseteq V$

Theorem L^2_0 may be expressed as a Hilbert direct sum of irreducible $\underbrace{\text{subrepresentations}}_{\text{(closed invariant subspace)}}$, each occurring w/ finite multiplicity: $\exists$ subreps $V_j \subseteq L^2_0$ s.t.

(i) $L^2_0 = \widehat{\bigoplus} V_j := \text{closure}(\bigoplus V_j)$,

(ii) $\forall j$, $\#\{k : V_j \cong V_k\} < \infty$

$\cong$: isomorphism of unitary representations of G.

(We'll see after that this implies $\{\text{cusp forms}\}$ is dense in L^2_0.)

Remark $\mathbb{R} \circlearrowright L^2(\mathbb{R})$ has no irreducible subspaces.

(idea: for $\xi \in \mathbb{R}$, the functions $\varphi_\xi(x) := e(\xi x)$ are $\mathbb{R}$-eigenfunctions: $\forall y \in \mathbb{R}$, $\varphi_\xi(x+y) = e(\xi y)\varphi_\xi(x)$. So the 1-dim'l spaces $\mathbb{C}\varphi_\xi$ are $\mathbb{R}$-invariant, irreducible. But $\varphi_\xi \notin L^2(\mathbb{R})$.)

Remark (i) is analogous to the fact ~~that~~

$$\{\text{trigonometric polynomials}\} \text{ is dense in } L^2(\mathbb{R}/\mathbb{Z}).$$

Proof Consider all subspaces V of L^2_0 s.t. $\exists V_j$: irreducible subreps s.t. $V = \widehat{\bigoplus} V_j$. The set of such V satisfies the hypotheses of Zorn's lemma, hence $\exists$ maximal such V. (Indeed, choose any max'l collection (V_j) of mutually orthogonal irreducible subreps, set $V := \widehat{\bigoplus} V_j$.) Goal $V = L^2_0$.

If not, then $V' := V^\perp \subseteq L^2_0$ is nonzero. To obtain a contradiction, it suffices to show that V' ($\subseteq L^2_0$) contains some irreducible subrep.

Let $0 \neq v \in V'$. By continuity of $G \circlearrowright L^2$, we may find $f \in C_c^\infty(G)$ s.t. $T_f v = v \star f \neq 0$.

(Indeed, $\exists$ neighborhood $1 \in \mathcal{U} \subseteq G$ s.t.

$$\|gv - v\| \leq \varepsilon \quad \forall g \in \mathcal{U}. \quad \text{Take } \varepsilon < \|v\|/2$$

and f supported in $\mathcal{U}$ with $\int f = 1$.)

We may arrange that f : $\mathbb{R}$-valued, $f(g^{-1}) = f(g)$.

$\Rightarrow T_f$: self-adjoint. $\left(T_f^* = T_{f^*},\ f^*(g) = \overline{f(g^{-1})}.\right)$

(check)

$\Rightarrow T_f$: compact, self-adjoint.

(previous Thm)

$T_f v \neq 0$, so $T_f|_{V'} \neq 0$.

By the spectral theorem, $\exists \lambda \neq 0$: eigenvalue for $T_f \circlearrowright V'$, with finite-dimensional eigenspace

$$0 \neq \underset{\circlearrowleft}{V'_\lambda} \subseteq V'.$$

T_f acts by λ

Let $0 \neq u \in V'_\lambda$ be such that $\dim(V'_\lambda \cap \langle Gu \rangle)$ is minimal.

Write $Gu :=$ G-orbit of u, $\langle Gu \rangle :=$ closure$(\operatorname{span}(Gu))$.

$\langle Gu \rangle =: W$: G-inv. closed subspace of V'

Claim W : irreducible subrepresentation of V'.

Pf of claim Suppose otherwise $\exists W_1 \subseteq W$: proper, $\neq 0$, closed invariant subspace.

Set $W_1' := W_1^\perp$ in W, so $W = W_1 \oplus W_1'$.

$$u = u_1 + u_1'$$

Note that W_1, W_1': subreps. Hence W_1, W_1': inv. by $T_f - \lambda$.

$$0 = (T_f - \lambda) u = \underbrace{(T_f - \lambda) u_1}_{\in W_1} + \underbrace{(T_f - \lambda) u_1'}_{\in W_1'}$$

$\Rightarrow (T_f - \lambda) u_1 = (T_f - \lambda) u_1' = 0$, hence $u_1, u_1' \in V'_\lambda$. Check u_1, u_1': nonzero

But then $V'_\lambda \cap \langle G u_1 \rangle \subsetneq V'_\lambda \cap \langle G u \rangle$, contrary to minimality.